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SOME PROBLEMS OF VIBRATION AND STABILITY OF SHELLS AND PLATES

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Abstract—Considered are problems of free and forced vibrations, static and dynamic stability and aeroelasticity of orthotropic shells and plates placed in a variable temperature field. It is assumed that the physical-mechanical characteristics of the material of the shell (plate) depend on the temperature.

It is shown that taking into account the dependence of the physical-mechanical properties of the material of the shell (plate) on temperature introduces essential qualitative and quantitative changes in the problem of vibration and stability.

1. BASIC RELATIONS AND EQUATIONS

LET us consider a shallow flexible orthotropic shell of positive Gaussian curvature and uniform thickness h, placed in a variable temperature field.

The shell is referred to an orthogonal curvilinear coordinate system α, β, γ so that the median surface of the non-deformed shell coincides with the coordinate surface $\gamma = 0$, the coordinate axis γ is directed towards the convexity, while the coordinate lines $\beta = \text{const}$, $\gamma = 0$ and $\alpha = \text{const}$, $\gamma = 0$ coincide with the curvature lines of the median surface. The shell material follows the generalized Hooke's law and at every point possesses three planes of elastic symmetry, parallel to the coordinate surfaces [1].

The shell temperature $T = T(\alpha, \beta, \gamma, t)$ is assumed to satisfy the thermal conductivity equation as well as the initial ($t = t_0$) and surface conditions.

By virtue of the temperature field variability there must be considered a change in physico-mechanical properties of the shell material (moduli of elasticity $E_\alpha = E_1(T)$, $E_\beta = E_2(T)$ modulus of shear $G_{\alpha\beta} = G_{12}(T)$, temperature expansion coefficients α_1, α_2 etc.), caused by the temperature change due to heating. It is assumed, however, that in the temperature variation range under consideration the shell material is elastic.

In deriving the basic equations, the following propositions are used:

(1) the Kirchhoff–Love hypothesis on non-deformable normals [2];
(2) the Neumann hypothesis on the absence of shears in an infinitely small shell element in the case of temperature variation, generalized for orthotropic material [3];
(3) the Karman–Vlasov hypothesis allowing, in all the relations, to retain nonlinear elements containing the derivatives of normal displacement alone [4];
(4) generally known simplifications of the theory of shallow shells [1, 2].

In view of the generalized Hooke's law and making use of the introduced hypotheses and assumptions, the following relations for stresses are obtained [1, 3, 5, 6]

$$\sigma_\alpha = B_{11}\varepsilon_1+B_{12}\varepsilon_2+\gamma(B_{11}\varkappa_1+B_{12}\varkappa_2)-\beta_1 T,$$
$$\sigma_\beta = B_{12}\varepsilon_1+B_{22}\varepsilon_2+\gamma(B_{12}\varkappa_1+B_{22}\varkappa_2)-\beta_2 T, \qquad (1.1)$$
$$\tau_{\alpha\beta} = B_{66}(\omega+2\gamma\tau),$$

where

$$B_{11} = \frac{E_1}{1-\nu_1\nu_2}, B_{22} = \frac{E_2}{1-\nu_1\nu_2}, B_{12} = B_{21} = \frac{\nu_2 E_1}{1-\nu_1\nu_2} = \frac{\nu_1 E_2}{1-\nu_1\nu_2}, B_{66} = G_{12},$$
$$\beta_i = B_{1i}\alpha_1+B_{2i}\alpha_2,$$

$u = u(\alpha,\beta,t)$, $v = v(\alpha,\beta,t)$, $w = w(\alpha,\beta,t)$ are the tangential and normal displacements of the points of the shell median surface; $\nu_1 = \nu_{\beta\alpha}$, $\nu_2 = \nu_{\alpha\beta}$—are the Poisson ratios; $k_1 = 1/R_1$, $k_2 = 1/R_2$ are the principal curvatures of the median surface; A, B are the coefficients of the first quadratic form, whereas the relative deformations ε_1, ε_2, ω and variations of the curvature $\varkappa_1$, $\varkappa_2$, τ are expressed by the relations

$$\varepsilon_1 = \frac{1}{A}\frac{\partial u}{\partial\alpha}+k_1 w+\frac{1}{2A^2}\left(\frac{\partial w}{\partial\alpha}\right)^2, \qquad \varepsilon_2 = \frac{1}{B}\frac{\partial v}{\partial\beta}+k_2 w+\frac{1}{2B^2}\left(\frac{\partial w}{\partial\beta}\right)^2,$$
$$\omega = \frac{1}{A}\frac{\partial v}{\partial\alpha}+\frac{1}{B}\frac{\partial u}{\partial\beta}+\frac{1}{AB}\frac{\partial w}{\partial\alpha}\frac{\partial w}{\partial\beta}, \qquad (1.2)$$
$$\varkappa_1 = -\frac{1}{A^2}\frac{\partial^2 w}{\partial\alpha^2}, \qquad \varkappa_2 = -\frac{1}{B^2}\frac{\partial^2 w}{\partial\beta^2}, \qquad \tau = -\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta}.$$

The equations of motion and deformation continuity for the case under consideration are of the form†

$$B\frac{\partial T_1}{\partial\alpha}+A\frac{\partial S}{\partial\beta} = m^*\frac{\partial^2 u}{\partial t^2}, \qquad B\frac{\partial S}{\partial\alpha}+A\frac{\partial T_2}{\partial\beta} = m^*\frac{\partial^2 v}{\partial t^2},$$

$$\frac{1}{A^2}\frac{\partial^2 M_1}{\partial\alpha^2}+\frac{2}{AB}\frac{\partial^2 H}{\partial\alpha\partial\beta}+\frac{1}{B^2}\frac{\partial^2 M_2}{\partial\beta^2}-k_1T_1-k_2T_2+\frac{1}{A^2}\frac{\partial}{\partial\alpha}\left(\frac{\partial w}{\partial\alpha}T_1\right)+\frac{1}{AB}\frac{\partial}{\partial\beta}\left(\frac{\partial w}{\partial\alpha}S\right) \qquad (1.3)$$
$$+\frac{1}{AB}\frac{\partial}{\partial\alpha}\left(\frac{\partial w}{\partial\beta}S\right)+\frac{1}{B^2}\frac{\partial}{\partial\beta}\left(\frac{\partial w}{\partial\beta}T_2\right) = -Z+m^*\frac{\partial^2 w}{\partial t^2}$$

(here m^* is the shell mass per unit area of the median surface),

$$\frac{B}{A}\frac{\partial^2\varepsilon_2}{\partial\alpha^2}-\frac{\partial^2\omega}{\partial\alpha\partial\beta}+\frac{A}{B}\frac{\partial^2\varepsilon_1}{\partial\beta^2}-\frac{B}{A}k_2\frac{\partial^2 w}{\partial\alpha^2}-\frac{A}{B}k_1\frac{\partial^2 w}{\partial\beta^2}-\frac{1}{AB}\left(\frac{\partial^2 w}{\partial\alpha\partial\beta}\right)^2+\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha^2}\frac{\partial^2 w}{\partial\beta^2} = 0. \qquad (1.4)$$

By choosing a proper expression for Z, fundamental equations for all the problems of interest to us are, as is known [7, 8], described by the system (1.3), (1.4).

For tangential forces ($T_\alpha = T_1$, $T_\beta = T_2$, $S_{\alpha\beta} = S_{\beta\alpha} = S$) and moments ($M_\alpha = M_1$, $M_\beta = M_2$, $H_{\alpha\beta} = H_{\beta\alpha} = H$) included in (1.3), we have [1, 5, 6]

† Later on, to simplify the presentation, the tangential components of an external surface load are assumed to be equal to zero ($X = 0, Y = 0$), although the case of $X \neq 0$ and $Y \neq 0$ does not introduce any principal changes into the solution of the problems under consideration.

$$
\begin{gathered}
T_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + K_{11}\varkappa_1 + K_{12}\varkappa_2 - C_{1T}, \\
T_2 = C_{12}\varepsilon_1 + C_{22}\varepsilon_2 + K_{12}\varkappa_1 + K_{22}\varkappa_2 - C_{2T}, \\
S = C_{66}\omega + 2K_{66}\tau,
\end{gathered} \tag{1.5}
$$

$$
\begin{gathered}
M_1 = K_{11}\varepsilon_1 + K_{12}\varepsilon_2 + D_{11}\varkappa_1 + D_{12}\varkappa_2 - K_{1T}, \\
M_2 = K_{12}\varepsilon_1 + K_{22}\varepsilon_2 + D_{12}\varkappa_1 + D_{22}\varkappa_2 - K_{2T}, \\
H = K_{66}\omega + 2D_{66}\tau,
\end{gathered} \tag{1.6}
$$

where the rigidities $C_{ij} = C_{ij}(\alpha, \beta, t)$, $K_{ij} = K_{ij}(\alpha, \beta, t)$, $D_{ij} = D_{ij}(\alpha, \beta, t)$, the temperature strains and moments $C_{iT} = C_{iT}(\alpha, \beta, t)$, $K_{iT} = K_{iT}(\alpha, \beta, t)$ are expressed by the formulae

$$
C_{ij} = \int_{-h/2}^{h/2} B_{ij}\,\mathrm{d}\gamma, \qquad K_{ij} = \int_{-h/2}^{h/2} B_{ij}\gamma\,\mathrm{d}\gamma, \qquad D_{ij} = \int_{-h/2}^{h/2} B_{ij}\gamma^2\,\mathrm{d}\gamma \qquad (i,j = 1,2,6),
$$

$$
C_{iT} = \int_{-h/2}^{h/2} \beta_i T\,\mathrm{d}\gamma, \qquad K_{iT} = \int_{-h/2}^{h/2} \beta_i T\gamma\,\mathrm{d}\gamma \qquad (i = 1,2).
$$

Substituting (1.2) and (1.5), (1.6) into (1.3), to determine the sought displacements u, v, w, the following system of nonlinear differential equations with variable coefficients is obtained

$$
\begin{gathered}
L_1(u) + L_2(v) + L_3(w) + Q_1(w,w) = \frac{1}{A}\frac{\partial C_{1T}}{\partial\alpha} + m^*\frac{\partial^2 u}{\partial t^2}, \\
L_1^*(v) + L_2^*(u) + L_3^*(w) + Q_1^*(w,w) = \frac{1}{B}\frac{\partial C_{2T}}{\partial\beta} + m^*\frac{\partial^2 v}{\partial t^2}, \\
L_4(u) + L_4^*(v) + (L_5 + L_5^*)(w) + Q_2(w,u) + Q_2^*(w,v) + Q_3(w,w) + Q_3^*(w,w) \\
= \frac{1}{A^2}\frac{\partial^2 K_{1T}}{\partial\alpha^2} + \frac{1}{B^2}\frac{\partial^2 K_{2T}}{\partial\beta^2} - (k_1 C_{1T} + k_2 C_{2T}) - Z + m^*\frac{\partial^2 w}{\partial t^2},
\end{gathered} \tag{1.7}
$$

where for linear (L_i) and nonlinear (Q_i) operators we have

$$
L_1() = \frac{1}{A^2}\frac{\partial}{\partial\alpha}\left[C_{11}\frac{\partial()}{\partial\alpha}\right] + \frac{1}{B^2}\frac{\partial}{\partial\beta}\left[C_{66}\frac{\partial()}{\partial\beta}\right],
$$

$$
L_2() = \frac{1}{AB}\left\{\frac{\partial}{\partial\alpha}\left[C_{12}\frac{\partial()}{\partial\beta}\right] + \frac{\partial}{\partial\beta}\left[C_{66}\frac{\partial()}{\partial\alpha}\right]\right\},
$$

$$
L_3() = -\frac{1}{A^3}\frac{\partial}{\partial\alpha}\left[K_{11}\frac{\partial^2()}{\partial\alpha^2}\right] - \frac{1}{AB^2}\left\{\frac{\partial}{\partial\alpha}\left[K_{12}\frac{\partial^2()}{\partial\beta^2}\right] + 2\frac{\partial}{\partial\beta}\left[K_{66}\frac{\partial^2()}{\partial\alpha\partial\beta}\right]\right\} + \frac{1}{A}\frac{\partial}{\partial\alpha}[(k_1C_{11} + k_2C_{12})()],
$$

$$
L_4() = \frac{1}{A^3}\frac{\partial^2}{\partial\alpha^2}\left[K_{11}\frac{\partial()}{\partial\alpha}\right] + \frac{1}{AB^2}\left\{\frac{\partial^2}{\partial\beta^2}\left[K_{12}\frac{\partial()}{\partial\alpha}\right] + 2\frac{\partial^2}{\partial\alpha\partial\beta}\left[K_{66}\frac{\partial()}{\partial\beta}\right]\right\} - \frac{1}{A}(k_1C_{11} + k_2C_{12})\frac{\partial()}{\partial\alpha},
$$

$$L_5(\,)=\frac{1}{A^2}\left\{\frac{\partial^2}{\partial\alpha^2}\left[-\frac{D_{11}}{A^2}\frac{\partial^2(\,)}{\partial\alpha^2}-\frac{D_{12}}{B^2}\frac{\partial^2(\,)}{\partial\beta^2}+(k_1K_{11}+k_2K_{12})(\,)\right]-\frac{2}{B^2}\frac{\partial^2}{\partial\alpha\partial\beta}\left[D_{66}\frac{\partial^2(\,)}{\partial\alpha\partial\beta}\right]\right.$$

$$\left.-\frac{\partial}{\partial\alpha}\left[C_{1T}\frac{\partial(\,)}{\partial\alpha}\right]\right\}+k_1\left[\frac{K_{11}}{A^2}\frac{\partial^2(\,)}{\partial\alpha^2}+\frac{K_{12}}{B^2}\frac{\partial^2(\,)}{\partial\beta^2}-(k_1C_{11}+k_2C_{12})(\,)\right],$$

$$Q_1(w,w)=\frac{1}{2AB^2}\left\{\frac{\partial}{\partial\alpha}\left[\frac{B^2}{A^2}C_{11}\left(\frac{\partial w}{\partial\alpha}\right)^2+C_{12}\left(\frac{\partial w}{\partial\beta}\right)^2\right]+2\frac{\partial}{\partial\beta}\left(C_{66}\frac{\partial w}{\partial\alpha}\frac{\partial w}{\partial\beta}\right)\right\},$$

$$Q_2[w,(\,)]=\frac{1}{AB^2}\left\{\frac{\partial}{\partial\alpha}\left[\frac{B^2}{A^2}C_{11}\frac{\partial w}{\partial\alpha}\frac{\partial(\,)}{\partial\alpha}+C_{66}\frac{\partial w}{\partial\beta}\frac{\partial(\,)}{\partial\beta}\right]+\frac{\partial}{\partial\beta}\left[C_{12}\frac{\partial w}{\partial\beta}\frac{\partial(\,)}{\partial\alpha}+C_{66}\frac{\partial w}{\partial\alpha}\frac{\partial(\,)}{\partial\beta}\right]\right\},$$

$$Q_3(w,w)=\frac{1}{2A^4}\frac{\partial}{\partial\alpha}\left[\left(C_{11}\frac{\partial w}{\partial\alpha}+\frac{\partial K_{11}}{\partial\alpha}\right)\left(\frac{\partial w}{\partial\alpha}\right)^2\right]+\frac{1}{2A^2}\left\{(k_1C_{11}+k_2C_{12})\left(\frac{\partial w}{\partial\alpha}\right)^2\right.$$

$$\left.+2w\frac{\partial}{\partial\alpha}\left[(k_1C_{11}+k_2C_{12})\frac{\partial w}{\partial\alpha}\right]\right\}+\frac{1}{2A^2B^2}\left\{\frac{\partial^2}{\partial\alpha^2}\left[K_{12}\left(\frac{\partial w}{\partial\beta}\right)^2\right]\right.$$

$$+2\frac{\partial^2}{\partial\alpha\partial\beta}\left(K_{66}\frac{\partial w}{\partial\alpha}\frac{\partial w}{\partial\beta}\right)+2\frac{\partial w}{\partial\alpha}\left[C_{66}\frac{\partial w}{\partial\beta}\frac{\partial^2 w}{\partial\alpha\partial\beta}+\frac{\partial}{\partial\beta}\left(C_{66}\frac{\partial w}{\partial\alpha}\frac{\partial w}{\partial\beta}\right)\right]$$

$$\left.+\frac{\partial}{\partial\alpha}\left[C_{12}\frac{\partial w}{\partial\alpha}\left(\frac{\partial w}{\partial\beta}\right)^2-2K_{12}\frac{\partial w}{\partial\alpha}\frac{\partial^2 w}{\partial\beta^2}\right]-4\frac{\partial}{\partial\beta}\left(K_{66}\frac{\partial w}{\partial\alpha}\frac{\partial^2 w}{\partial\alpha\partial\beta}\right)\right\}.$$

In (1.7) the operators L_i^* and Q_i^* are derived from the operators L_i and Q_i, replacing A by B, α by β; index 1 by 2 and index 2 by 1.

The original equations of the problem may also be derived by means of a mixed method, i.e. with the aid of the stress function $\varphi(\alpha,\beta,t)$ and by the displacement function w.

As usual, for a shallow shell, neglecting tangential forces of inertia and introducing the stress function by the formulae

$$T_1=\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2},\qquad T_2=\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2},\qquad S=-\frac{1}{AB}\frac{\partial^2\varphi}{\partial\alpha\partial\beta}\tag{1.8}$$

the first two equations of equilibrium (1.3) are identically satisfied.

Substituting (1.8) into (1.5) and solving a system of algebraic equations with respect to ε_1, ε_2 and ω, one obtains

$$\varepsilon_1=A_{11}\left(\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2}+C_{1T}\right)+A_{12}\left(\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2}+C_{2T}\right)+\frac{d_{11}}{A^2}\frac{\partial^2 w}{\partial\alpha^2}+\frac{d_{12}}{B^2}\frac{\partial^2 w}{\partial\beta^2},$$

$$\varepsilon_2=A_{12}\left(\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2}+C_{1T}\right)+A_{22}\left(\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2}+C_{2T}\right)+\frac{d_{21}}{A^2}\frac{\partial^2 w}{\partial\alpha^2}+\frac{d_{22}}{B^2}\frac{\partial^2 w}{\partial\beta^2},\tag{1.9}$$

$$\omega=-\frac{A_{66}}{AB}\frac{\partial^2\varphi}{\partial\alpha\partial\beta}+2\frac{d_{66}}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta},$$

where the notation used is

$$A_{11} = \frac{C_{22}}{\Omega}, \quad A_{12} = -\frac{C_{12}}{\Omega}, \quad A_{22} = \frac{C_{11}}{\Omega}, \quad A_{66} = \frac{1}{C_{66}}, \quad \Omega = C_{11}C_{22} - C_{12}^2,$$

$$d_{11} = \frac{C_{22}K_{11} - C_{12}K_{12}}{\Omega}, \quad d_{12} = \frac{C_{22}K_{12} - K_{22}C_{12}}{\Omega}, \quad d_{21} = \frac{C_{11}K_{12} - C_{12}K_{11}}{\Omega},$$

$$d_{22} = \frac{C_{11}K_{22} - C_{12}K_{12}}{\Omega}, \quad d_{66} = \frac{K_{66}}{C_{66}},$$

Substituting (1.2), (1.9) into (1.6) one obtains

$$M_1 = d_{11}\left(\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} + C_{1T}\right) + d_{21}\left(\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} + C_{2T}\right) + \frac{1}{A^2}(D_{11}^* - D_{11})\frac{\partial^2 w}{\partial\alpha^2} + \frac{1}{B^2}(D_{12}^* - D_{12})\frac{\partial^2 w}{\partial\beta^2} - K_{1T},$$

$$M_2 = d_{12}\left(\frac{1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} + C_{1T}\right) + d_{22}\left(\frac{1}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} + C_{2T}\right) + \frac{1}{A^2}(D_{12}^* - D_{12})\frac{\partial^2 w}{\partial\alpha^2} + \frac{1}{B^2}(D_{22}^* - D_{22})\frac{\partial^2 w}{\partial\beta^2} - K_{2T}, \quad (1.10)$$

$$H = -\frac{d_{66}}{AB}\frac{\partial^2\varphi}{\partial\alpha\partial\beta} + \frac{2}{AB}(D_{66}^* - D_{66})\frac{\partial^2 w}{\partial\alpha\partial\beta}.$$

Here

$$D_{11}^* = K_{11}d_{11} + K_{12}d_{21}, \quad D_{12}^* = K_{11}d_{12} + K_{12}d_{22} = K_{12}d_{11} + K_{22}d_{21},$$
$$D_{22}^* = K_{12}d_{12} + K_{22}d_{22}, \quad D_{66}^* = K_{66}d_{66}.$$

Substituting the values of deformations, forces and moments from (1.8)–(1.10) into the third equation of motion (1.3) and into the equation of deformation continuity (1.4), to determine the stress function and normal displacement, one obtains the following fundamental system of nonlinear differential equations of the problem

$$(L_6 + L_6^*)(\varphi) + (L_7 + L_7^*)(w) + Q_4(w, \varphi) = \frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}(K_{1T} - C_{1T}d_{11} - C_{2T}d_{21}) + \frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}(K_{2T} - C_{1T}d_{12} - C_{2T}d_{22}) - Z + m^*\frac{\partial^2 w}{\partial t^2}, \quad (1.11)$$

$$(L_8 + L_8^*)(\varphi) + (L_9 + L_9^*)(w) + \tfrac{1}{2}Q_4(w, w) = -\frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}(C_{1T}A_{12} + C_{2T}A_{22}) - \frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}(C_{1T}A_{11} + C_{2T}A_{21}),$$

where

$$L_6(\,) = \frac{1}{A^2}\left\{\frac{\partial^2}{\partial\alpha^2}\left[\frac{d_{21}}{A^2}\frac{\partial^2(\,)}{\partial\alpha^2} + \frac{d_{11}}{B^2}\frac{\partial^2(\,)}{\partial\beta^2} - k_2(\,)\right] - \frac{1}{B^2}\frac{\partial^2}{\partial\alpha\partial\beta}\left[d_{66}\frac{\partial^2(\,)}{\partial\alpha\partial\beta}\right]\right\},$$

$$L_7(\,) = \frac{1}{A^2}\left\{\frac{\partial^2}{\partial\alpha^2}\left[\frac{D_{11}^* - D_{11}}{A^2}\frac{\partial^2(\,)}{\partial\alpha^2} + \frac{D_{12}^* - D_{12}}{B^2}\frac{\partial^2(\,)}{\partial\beta^2}\right] + \frac{2}{B^2}\frac{\partial^2}{\partial\alpha\partial\beta}\left[(D_{66}^* - D_{66})\frac{\partial^2 w}{\partial\alpha\partial\beta}\right]\right\},$$

$$L_8(\,) = \frac{1}{A^2}\left\{\frac{\partial^2}{\partial\alpha^2}\left[\frac{A_{22}}{A^2}\frac{\partial^2(\,)}{\partial\alpha^2} + \frac{A_{12}}{B^2}\frac{\partial^2(\,)}{\partial\beta^2}\right] + \frac{1}{2B^2}\frac{\partial^2}{\partial\alpha\partial\beta}\left[A_{66}\frac{\partial^2(\,)}{\partial\alpha\partial\beta}\right]\right\},$$

$$L_9(\,) = \frac{1}{A^2}\left\{\frac{\partial^2}{\partial\alpha^2}\left[\frac{d_{21}}{A^2}\frac{\partial^2(\,)}{\partial\alpha^2} + \frac{d_{22}}{B^2}\frac{\partial^2(\,)}{\partial\beta^2} - k_2(\,)\right] - \frac{1}{B^2}\frac{\partial^2}{\partial\alpha\partial\beta}\left(d_{66}\frac{\partial^2 w}{\partial\alpha\partial\beta}\right)\right\},$$

$$Q_4[w,(\,)] = \frac{1}{A^2B^2}\left[\frac{\partial^2 w}{\partial\alpha^2}\frac{\partial^2(\,)}{\partial\beta^2} - 2\frac{\partial^2 w}{\partial\alpha\partial\beta}\frac{\partial^2(\,)}{\partial\alpha\partial\beta} + \frac{\partial^2 w}{\partial\beta^2}\frac{\partial^2(\,)}{\partial\alpha^2}\right],$$

and the operators with asterisks are determined as before.

Thus, the solution of the problems under consideration is reduced to the integration of the system of differential equations (1.7) or (1.11). The integration functions (in a particular case—the integration constants) included in the integrals of these equations are determined from the familiar boundary conditions.

In all the problems to be considered later the shell temperature is assumed to be independent of the coordinates of the point and to vary with time only $T = T(t)$, $\partial T/\partial\alpha = \partial T/\partial\beta = \partial T/\partial\gamma = 0$.

2. FREE TRANSVERSE VIBRATION OF A SHELL RECTANGULAR IN PLAN

Let us consider, in a linear formulation, free transverse vibrations of a rectangular in plan ($a \times b$), hinge-supported throughout the contour, shallow orthotropic shell in a time-dependent temperature field [9].

For the case under consideration in (1.11) the nonlinear operator should be omitted, and one should assume

$$Z = K_{ij} = d_{ij} = D_{ij}^* = \frac{\partial A_{ij}}{\partial\alpha} = \frac{\partial A_{ij}}{\partial\beta} = \frac{\partial D_{ij}}{\partial\alpha} = \frac{\partial D_{ij}}{\partial\beta} = 0.$$

Then we obtain

$$\begin{aligned} &\frac{D_{11}}{A^4}\frac{\partial^2 w}{\partial\alpha^4} + 2\frac{D_{12} + 2D_{66}}{A^2B^2}\frac{\partial^4 w}{\partial\alpha^2\partial\beta^2} + \frac{D_{22}}{B^4}\frac{\partial^4 w}{\partial\beta^4} + \frac{k_2}{A^2}\frac{\partial^2\varphi}{\partial\alpha^2} + \frac{k_1}{B^2}\frac{\partial^2\varphi}{\partial\beta^2} = -m^*\frac{\partial^2 w}{\partial t^2}, \\ &\frac{A_{22}}{A^4}\frac{\partial^4\varphi}{\partial\alpha^4} + \frac{2A_{12} + A_{66}}{A^2B^2}\frac{\partial^4\varphi}{\partial\alpha^2\partial\beta^2} + \frac{A_{11}}{B^4}\frac{\partial^4\varphi}{\partial\beta^4} - \frac{k_2}{A^2}\frac{\partial^2 w}{\partial\alpha^2} - \frac{k_1}{B^2}\frac{\partial^2 w}{\partial\beta^2} = 0. \end{aligned} \tag{2.1}$$

The boundary and initial conditions will be

$$\begin{aligned} &w = T_1 = M_1 = 0 \quad \text{when} \quad \beta = 0 \ \text{and} \ \beta = b, \\ &w = T_2 = M_2 = 0 \quad \text{when} \quad \alpha = 0 \ \text{and} \ \alpha = a, \end{aligned} \tag{2.2}$$

$$w = w_0 \quad \text{and} \quad \partial w/\partial t = w_0' \quad \text{when} \quad t = 0, \ \alpha = a/2, \ \beta = b/2. \tag{2.3}$$

The boundary conditions (2.2) are apparently satisfied, assuming

$$
\begin{aligned}
w &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f_{nm}(t) \sin(\pi n\alpha/a) \sin(\pi m\beta/b), \\
\varphi &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} F_{nm}(t) \sin(\pi n\alpha/a) \sin(\pi m\beta/b).
\end{aligned}
\tag{2.4}
$$

Substituting w and φ from (2.4) into the initial equations (2.1), to determine the unknown $f_{nm}(t)$ and $F_{nm}(t)$ we obtain

$$
F_{nm}(t) = -\left(\frac{ABab}{\pi}\right)^2 \frac{A^2a^2m^2k_1 + B^2b^2n^2k_2}{B^4b^4n^4A_{22} + A^2B^2a^2b^2n^2m^2(2A_{12} + A_{66}) + A^4a^4m^4A_{11}} f_{nm}(t)
$$

where the function $f_{nm}(t)$ should satisfy the following ordinary differential equation of the second order with variable coefficients

$$
\ddot{f}_{nm}(t) + \delta^2\psi_{nm}(t) f_{nm}(t) = 0. \tag{2.5}
$$

The notation used here is

$$
\delta^2 = \frac{B_{11}(t_0)h}{12m^*gAa}\pi^2,
$$

$$
\begin{aligned}
\psi_{nm}(t) = \frac{B_{11}(t)}{B_{11}(t_0)} \frac{g}{Aa} \Bigg\{ & \left(\frac{\pi h}{Aa}\right)^2 \left[n^4 + 2\left(v_2 + 2\frac{B_{66}(t)}{B_{11}(t)}\right)\left(\frac{Aanm}{Bb}\right)^2 + \frac{B_{22}(t)}{B_{11}(t)}\left(\frac{Aam}{Bb}\right)^4 \right] \\
& + 12(1 - v_1v_2)\left(\frac{Aa}{\pi}\right)^2 (A^2a^2m^2k_1 + B^2b^2n^2k_2)^2 \Bigg[\frac{B_{11}(t)}{B_{22}(t)}(Bbn)^4 \\
& + \left(\frac{E_1(t)}{G_{12}(t)} - 2v_1\right)(ABabnm)^2 + (Aam)^4 \Bigg]^{-1} \Bigg\},
\end{aligned}
$$

and dots, as usual, denote the time derivatives.

For many applied problems of shell theory the dimensionless value δ^2 is a large number ($\delta^2 \gg 1$). Accordingly, the solution for the equation (2.5) is found by the method of asymptotic integration [1, 2, 9].

The integrals of the equation (2.5) are to be sought in the form (later, wherever possible, the indices n and m are omitted)·

$$
f(t) = \Phi(t;\delta)e^{\delta\omega(t)}, \tag{2.6}
$$

where $\Phi(t;\delta) = \Phi_0(t) + \delta^{-1}\Phi_1(t) + \delta^{-2}\Phi_2(t) + \ldots$ is the intensity function, and $\omega(t)$ is the function of variability.

Substituting (2.6) into (2.5) and properly grouping the terms, we obtain

$$
\begin{aligned}
& e^{\delta\omega}\{\delta^2\Phi_0(\psi + \dot{\omega}^2) + \delta[2\dot{\Phi}_0\dot{\omega} + \Phi_0\ddot{\omega} + \Phi_1(\psi + \dot{\omega}^2)] \\
& \qquad + \sum_{j=0}^{N} \delta^{-j}[\ddot{\Phi}_j + 2\dot{\Phi}_{j+1}\dot{\omega} + \Phi_{j+1}\ddot{\omega} + \Phi_{j+2}(\psi + \dot{\omega}^2)]\} = 0,
\end{aligned}
$$

whence, reducing by $e^{\delta\omega}$ and equating coefficients of various powers of δ to zero, under the condition $\Phi_0 \neq 0$ we obtain

$$\psi + \dot{\omega}^2 = 0, \tag{2.7}$$

$$2\dot{\Phi}_0\dot{\omega} + \Phi_0\ddot{\omega} = 0, \tag{2.8}$$

as well as the recurrent equations

$$\ddot{\Phi}_j + 2\dot{\Phi}_{j+1}\dot{\omega} + \Phi_{j+1}\ddot{\omega} = 0 \qquad (j = 0, 1, 2, \dots). \tag{2.9}$$

Solving (2.7), for the function of variability we have

$$\omega_1 = i\int\sqrt{\psi}\,dt + C_1', \qquad \omega_2 = -i\int\sqrt{\psi}\,dt + C_2'. \tag{2.10}$$

Substituting (2.10) into (2.8), one finds the intensity function in zero-th approximation

$$\Phi_0 = C_1''\psi^{-\frac{1}{4}},$$

and each successive approximation is found by integration (2.9) at successive values $j = 1, 2, \dots$.

Thus, in zero-th approximation, for the function f one obtains

$$f = \psi^{-\frac{1}{4}}[C_1 \exp(i\delta\int\sqrt{\psi}\,dt) + C_2 \exp(-i\delta\int\sqrt{\psi}\,dt)], \tag{2.11}$$

where the values of the integration constants C_1 and C_2 are readily determined from the conditions (2.3), using the representation (2.4).

Examination of (2.11) shows that in a number of cases the zero-th approximation provides a quite satisfactory accuracy of the solution.

If the expansions (2.4) are restricted to the first terms only, it is readily seen that at $\psi_{11}(t) = \text{const} = 1$ for the function f in the zero-th approximation one obtains

$$f = C_1 \exp(i\delta t) + C_2 \exp(-i\delta t)$$

which coincides with an exact solution of the equation $\ddot{f} + \delta^2 f = 0$.

In the case of a linear attenuation, the differential equation of motion (2.5) assumes the form

$$\ddot{f}(t) + 2\varepsilon\dot{f}(t) + \delta^2\psi(t)f(t) = 0, \tag{2.12}$$

where ε is the constant coefficient of attenuation.

As is known, by proper replacement of the constant the equation (2.12) may be reduced to an equation of the form (2.5), and consequently, it is possible to find solution for the problem of vibration of a plate being heated, accounting for linear attenuation.

In a special case let us consider free transverse vibration of a rectangular plate ($k_1 = k_2 = 0$), hinge-supported throughout the contour [10].

The mean temperature of the plate is assumed to vary according to the law

$$T = \frac{T_{\max}}{t_1}t \quad (0 \le t \le t_1), \qquad T = T_{\max} \quad (t > t_1),$$

where $T_{\max}$ is the maximal temperature of the plate, t_1 is the time, necessary to reach the constant temperature $T_{\max}$.

It follows from experimental investigations [11, 12] that the moduli of elasticity of various materials vary with temperature either almost linearly

$$E_i = E_i^0(1 - \lambda' T), \qquad E_i^0 = E_i(t_0) = \text{const},$$

or along a curve, which is readily approximated by a power (generally quadratic) dependence.

The coefficients of elasticity B_{ij} are approximately assumed to depend linearly on temperature

$$B_{ij} = B_{ij}^0(1 - \lambda'' T), \qquad B_{ij}^0 = B_{ij}(t_0) = \text{const.}$$

Then from (2.5), restricting ourselves to the principal form of vibration ($n = m = 1$), we obtain [10]

$$\ddot{f} + c^2(1 - \lambda t)f = 0, \tag{2.13}$$

where

$$c^2 = \frac{B_{11}^0 h^3}{12m^*}\left(\frac{\pi}{a}\right)^4\left[1 + 2\frac{a^2}{b^2}\left(\nu_2 + 2\frac{B_{66}^0}{B_{11}^0}\right) + \frac{a^4}{b^4}\frac{\nu_2}{\nu_1}\right] \approx \text{const}, \qquad \lambda = \frac{\lambda'' T_{\max}}{t_1},$$

and from the physical meaning of the problem it follows that $1 - \lambda t > 0$.

Equation (2.13) can be reduced to the Bessel equation and may be integrated exactly. Performing integration, substituting f into (2.4), and determining the values of the integration constants by initial conditions (2.3), we finally obtain

$$w = \frac{\pi}{\sqrt{3}}(\xi_0^2\xi)^{\frac{1}{3}}\Big\{w_0[J_{\frac{2}{3}}(\xi_0)T_{\frac{1}{3}}(\xi) + J_{-\frac{2}{3}}(\xi_0)J_{-\frac{1}{3}}(\xi)] - \frac{w_0'}{c}[J_{-\frac{1}{3}}(\xi_0)J_{\frac{1}{3}}(\xi) - J_{\frac{1}{3}}(\xi_0)T_{-\frac{1}{3}}(\xi)]\Big\}\sin\frac{\pi\alpha}{a}\,.\,\sin\frac{\pi\beta}{b}, \tag{2.14}$$

where, for the sake of brevity of presentation, a new dimensionless variable ξ is introduced by the formula

$$\xi = \frac{2c}{3\lambda}(1 - \lambda t)^{\frac{3}{2}}, \qquad \text{and} \quad \xi_0 = \xi_{t=0} = \frac{2c}{3\lambda}.$$

It is possible to show by simple analysis that with an increase of t both amplitude and conditional period of vibration of a plate being heated increase. The converse phenomenon may be observed when the plate becomes cooler, that is with an increase of its rigidity.

When considering the linear attenuation, instead of (2.13) we have the equation

$$\ddot{f} + 2\varepsilon\dot{f} + c^2(1 - \lambda t)f = 0$$

whose solution in terms of the deflection function w is

$$w = e^{-\varepsilon t}\sqrt{[c^2(1 - \lambda t) - \varepsilon^2]}\left\{C_1 J_{\frac{1}{3}}\left[\frac{2c}{3\lambda}(1 - \lambda t - \varepsilon^2/c^2)^{\frac{3}{2}}\right] + C_2 J_{-\frac{1}{3}}\left[\frac{2c}{3\lambda}(1 - \lambda t - \varepsilon^2/c^2)^{\frac{3}{2}}\right]\right\}\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}. \tag{2.15}$$

It should be noted, without going into elementary details, that in the final solution of the problem, that is in the formula for w, along with certain modifications in the argument of the Bessel functions, there is an attenuating factor $\exp(-\varepsilon t)$ which, depending upon the relative values of ε, c, λ, may significantly change the type of plate vibrations. For example, unlike the case for $\varepsilon = 0$, at high values of ε, the increase of temperature may be accompanied by a decrease of the plate vibration amplitude.

For many applied problems the arguments of the Bessel functions, included in (2.14), (2.15), possess high values. Making use of the familiar asymptotic formula

$$J_\nu(\alpha) = \sqrt{\frac{2}{\pi\alpha}}\cos\left(\alpha - \frac{\pi\nu}{2} - \frac{\pi}{4}\right),$$

from (2.14) one obtains

$$w = (\xi/\xi_0)^{-\frac{1}{4}}\left[w_0\cos(\xi_0 - \xi) + \frac{w_0'}{c}\sin(\xi_0 - \xi)\right]\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}. \tag{2.16}$$

It is evident that the approximate solution obtained in zero-th approximation by asymptotic integration of the equation (2.13), coincides with the exact solution (2.14), provided the Bessel functions included in it are represented asymptotically, that is—by (2.16).

3. FORCED TRANSVERSE VIBRATIONS OF A SHELL RECTANGULAR IN PLAN

The solution of the problem of forced vibrations of a shell being heated, considered in Section 2, is reduced to the integration of the differential equation

$$\ddot{f}_{nm}(t) + \delta^2\psi_{nm}(t)f_{nm}(t) = \frac{1}{m^*}Z'_{nm} \tag{3.1}$$

where Z'_{nm} denote the coefficients of the expansion of the load Z' into the double trigonometric series

$$Z' = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Z'_{nm}\sin(\pi n\alpha/a)\sin(\pi m\beta/b). \tag{3.2}$$

It is assumed that the expansion coefficients Z'_{nm} satisfy the known conditions and in their turn may be expanded in a trigonometric absolutely converging series of the form (3.2).

Then, as is known [2], the right hand side of the equation (3.1) may be represented in the form (later, wherever possible, indices n and m are omitted)

$$\frac{1}{m^*}Z' = \sum_{j=1}^{\infty}\delta^{q_j}\chi_j(t)e^{\delta_j\Omega_j(t)}, \tag{3.3}$$

where $\chi_j(t)$ and $\Omega_j(t)$ denote bounded complex functions differentiable the necessary number of times, whose method of determination is developed in monograph [2].

Making use of the linearity of equation (3.1), the problem may be simplified, assuming that on the right-hand side of this equation there is one of the terms of expansion (3.3), that is restricting ourselves to considering the equation

$$\ddot{f}(t) + \delta^2\psi(t)f(t) = \delta^q\chi(t)e^{\delta\Omega(t)}. \tag{3.4}$$

The solution for (3.4) is to be found in the form

$$f(t) = \delta^p\Phi(t;\delta)e^{\delta\Omega(t)}, \tag{3.5}$$

considering, as in (2.6), that the intensity function $\Phi(t;\delta)$ may be represented in the form of an asymptotic series

$$\Phi(t;\delta) = \Phi_0(t) + \delta^{-1}\Phi_1(t) + \delta^{-2}\Phi_2(t) + \dots \tag{3.6}$$

Substituting (3.6) into (2.5) and then the obtained value of $f(t)$ into (3.4), reducing the common exponential factor $e^{\delta\Omega}$ and making use of the condition $\Phi_0(t) \neq 0$, the value of p is found.

Then, equating the coefficients of all other powers of δ to zero, an infinite system of equations is obtained, from which the functions $\Phi_0, \Phi_1, \dots$, are subsequently determined.

Let us apply the above method to the case, when the right-hand side of equation (3.1) is a periodic function, whose frequency is expressed by fractions of a free vibration frequency of the same shell at initial temperature, that is

$$\frac{1}{m^*}Z'_{nm} = c\sin(\delta\sqrt{[\psi(t_0)]}\vartheta^* t) = c\sin(\delta\vartheta t).$$

Making use of the representation $\sin(\delta\vartheta t) = -{}^{i}/2(e^{i\delta\vartheta t} - e^{-i\delta\vartheta t})$, equation (3.1) may be rewritten as follows

$$\ddot{f} + \delta^2\psi f = -\frac{ci}{2}(e^{i\delta\vartheta t} - e^{-i\delta\vartheta t}). \tag{3.7}$$

A particular solution $f_p(t)$ of the equation (3.7), corresponding to its right hand side, is found by the above method.

For $\psi \neq \vartheta^2$ the following expression is obtained

$$f_p(t) = 2\delta^{-2}[i(\Phi_0 + \delta^{-2}\Phi_2 + \delta^{-4}\Phi_1 + \cdots)\sin(\delta\vartheta t)$$
$$+\delta^{-1}(\Phi_1 + \delta^{-2}\Phi_3 + \delta^{-4}\Phi_5 + \cdots)\cos(\delta\vartheta t)],$$

where

$$\Phi_0 = -\frac{ci}{2(\psi - \vartheta^2)}, \qquad \Phi_1 = -\frac{2i\vartheta\dot{\Phi}_0}{\psi - \vartheta^2},$$

and each subsequent function Φ_j, by means of the previous known functions, is determined by the following recurrence relation

$$\Phi_{j+2} = -\frac{\ddot{\Phi}_j + 2i\vartheta\dot{\Phi}_{j+1}}{\psi - \vartheta^2} \qquad (j = 0, 1, 2, \dots).$$

For the case of $\psi = \vartheta^2$ we obtain

$$f_p(t) = -\frac{c}{2\vartheta\delta}t\cos(\delta\vartheta t),$$

which coincides with the exact particular solution for the classic equation of resonance vibrations.

Thus the solution for the problem of forced vibrations of a plate being heated is obtained by superposition of (2.6) and (3.5). In this case the integration constants, included in (2.6), are found from the initial conditions of the problem.

In the case when the initial conditions of the problem contain the large parameter δ and can be represented in the form

$$f(t_0) = \Phi_0^* + \delta^{-1}\Phi_1^* + \delta^{-2}\Phi_2^* + \cdots, \qquad \dot{f}(t_0) = \Phi_0^{**} + \delta^{-1}\Phi_1^{**} + \delta^{-2}\Phi_2^{**} + \cdots \tag{3.8}$$

the integration constants c_j' and c_j'' $(j = 0, 1, 2, \ldots)$ entering the general solution of equation (3.4), will be found from the independent initial equations, written for the j-th approximation.

If the large parameter δ is absent in the initial conditions, when determining the values of the integration constants in (3.8), only the functions Φ_0^* and Φ_0^{**} should be taken different from zero and equal to the given initial functions, while the remaining functions Φ_j^*, Φ_j^{**} $(j = 1, 2, \ldots)$ should be assumed equal to zero.

4. STATIC STABILITY OF A FLEXIBLE RECTANGULAR PLATE WITH INITIAL IMPERFECTIONS

Let us consider the problem of static stability of a hinge-supported, heated, flexible rectangular $(a \times b)$ plate with initial imperfections $w_1 = w_1(\alpha, \beta)$.

In the initial relations and equations presented in Section 1, assuming $A = B = 1$, $k_1 = k_2 = 0$ and making use of the familiar method of considering the initial imperfections [4], the basic system of differential equations (1.11) may be rewritten in the following form

$$D_{11}\frac{\partial^4 w}{\partial\alpha^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial\alpha^2\partial\beta^2}+D_{22}\frac{\partial^4 w}{\partial\beta^4}-\frac{\partial^2\varphi}{\partial\alpha^2}\frac{\partial^2(w+w_1)}{\partial\beta^2}$$
$$+2\frac{\partial^2\varphi}{\partial\alpha\partial\beta}\frac{\partial^2(w+w_1)}{\partial\alpha\partial\beta}-\frac{\partial^2\varphi}{\partial\beta^2}\frac{\partial^2(w+w_1)}{\partial\alpha^2}=Z, \tag{4.1}$$

$$A_{22}\frac{\partial^4\varphi}{\partial\alpha^4}+(A_{66}+2A_{12})\frac{\partial^4\varphi}{\partial\alpha^2\partial\beta^2}+A_{11}\frac{\partial^4\varphi}{\partial\beta^4}+\frac{\partial^2(w+w_1)}{\partial\alpha^2}\frac{\partial^2(w+w_1)}{\partial\beta^2}-\left[\frac{\partial^2(w+w_1)}{\partial\alpha\partial\beta}\right]^2$$
$$=\frac{\partial^2 w_1}{\partial\alpha^2}\frac{\partial^2 w_1}{\partial\beta^2}-\left(\frac{\partial^2 w_1}{\partial\alpha\partial\beta}\right)^2.$$

Let us assume that the initial imperfections can be represented in the form

$$w_1 = f_0\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b} \tag{4.2}$$

and the average values of tensile forces, acting on the plate edges $\alpha = 0$, $\alpha = a$ and $\beta = 0$, $\beta = b$ are T_1^0 and T_2^0, respectively.

The conditions of hinge support will be satisfied if the deflection function w is taken in the form

$$w = f\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}. \tag{4.3}$$

Inserting (4.2) and (4.3) into the second equation of (4.1), for determination of the stress function (φ) we obtain the differential equation

$$A_{22}\frac{\partial^4\varphi}{\partial\alpha^4}+(A_{66}+2A_{12})\frac{\partial^4\varphi}{\partial\alpha^2\partial\beta^2}+A_{11}\frac{\partial^4\varphi}{\partial\beta^4}=\frac{\pi^4(f+2f_0)f}{2a^2b^2}\left(\cos\frac{2\pi\alpha}{a}+\cos\frac{2\pi\beta}{b}\right),$$

whence

$$\varphi = \frac{T_1^0\beta^2}{2}+\frac{T_2^0\alpha^2}{2}+\frac{f(f+2f_0)}{32}\left(\frac{a^2}{A_{22}b^2}\cos\frac{2\pi\alpha}{a}+\frac{b^2}{A_{11}a^2}\cos\frac{2\pi\beta}{b}\right). \tag{4.4}$$

Let us calculate the mutual approach Δu of the edges $\alpha = 0$, $\alpha = a$ and the mutual approach Δv of the edges $\beta = 0$, $\beta = b$.

Relations (1.2) for a plate with initial imperfections assume the form

$$\varepsilon_1 = \frac{\partial u}{\partial\alpha}+\frac{1}{2}\left[\frac{\partial(w+w_1)}{\partial\alpha}\right]^2-\frac{1}{2}\left(\frac{\partial w_1}{\partial\alpha}\right)^2,$$

$$\varepsilon_2 = \frac{\partial v}{\partial\beta}+\frac{1}{2}\left[\frac{\partial(w+w_1)}{\partial\beta}\right]^2-\frac{1}{2}\left(\frac{\partial w_1}{\partial\beta}\right)^2,$$

whence, considering (1.9) and (4.4), we obtain

$$\begin{aligned}\Delta u &= -\int_0^a \frac{\partial u}{\partial\alpha}\,\mathrm{d}\alpha = -A_{11}a(T_1^0+C_{1T})-A_{12}a(T_2^0+C_{2T})+\frac{f(f+2f_0)\pi^2}{8a},\\ \Delta v &= -\int_0^b \frac{\partial v}{\partial\beta}\,\mathrm{d}\beta = -A_{12}b(T_1^0+C_{1T})-A_{22}b(T_2^0+C_{2T})+\frac{f(f+2f_0)\pi^2}{8b}.\end{aligned} \tag{4.5}$$

In the case of non-approach of the edges we have $\Delta u = \Delta v = 0$.

Then, from (4.5) for T_1^0 and T_2^0 we obtain

$$\begin{aligned}T_1^0 &= \frac{f(f+2f_0)\pi^2}{8}\left(\frac{C_{11}}{a^2}+\frac{C_{12}}{b^2}\right)-C_{1T},\\ T_2^0 &= \frac{f(f+2f_0)\pi^2}{8}\left(\frac{C_{22}}{b^2}+\frac{C_{12}}{a^2}\right)-C_{2T}.\end{aligned} \tag{4.6}$$

As far as the first equation of (4.1) is concerned, its solution is found by the method of Bubnov–Galerkin. Then we will obtain

$$\int_0^a\int_0^b \left\{D_{11}\frac{\partial^4 w}{\partial\alpha^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial\alpha^2\partial\beta^2}+D_{22}\frac{\partial^4 w}{\partial\beta^4}\right.$$

$$\left.-\frac{\partial^2\varphi}{\partial\alpha^2}\frac{\partial^2(w+w_1)}{\partial\beta^2}+2\frac{\partial^2\varphi}{\partial\alpha\partial\beta}\frac{\partial^2(w+w_1)}{\partial\alpha\partial\beta}-\frac{\partial^2\varphi}{\partial\beta^2}\frac{\partial^2(w+w_1)}{\partial\alpha^2}-Z\right\}\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}\,\mathrm{d}\alpha\,\mathrm{d}\beta = 0.$$

Substituting here the values of w_1, w and φ from (4.2), (4.3) and (4.4), considering (4.6) and passing to the dimensionless deflections $\zeta = f/h$ and $\zeta_0 = f_0/h$, we obtain

$$\zeta^3+3\zeta_0\zeta^2+2\left(\zeta_0^2+\frac{\Delta_2}{\Delta_1}\right)\zeta-\frac{16\Delta_3\zeta_0+q}{\Delta_1} = 0, \tag{4.7}$$

where

$$\Delta_1 = (3-\nu_1\nu_2)\left(1+\frac{a^4}{b^4}\frac{E_2}{E_1}\right)+4\nu_2\frac{a^2}{b^2},$$

$$\Delta_2 = \frac{2}{3}\left[1+2\left(\nu_2+2\frac{B_{66}}{B_{11}}\right)\frac{a^2}{b^2}+\frac{E_2}{E_1}\frac{a^4}{b^4}-12\Delta_3\right],$$

$$\Delta_3 = \frac{a^2T}{\pi^2h^2}\left[\alpha_1+\nu_2\alpha_2+\frac{a^2}{b^2}\left(\nu_2\alpha_1+\frac{E_2}{E_1}\alpha_2\right)\right],$$

$$q = \frac{64a^3}{\pi^4B_{11}bh^4}\int_0^a\int_0^b Z\sin\frac{\pi\alpha}{a}\sin\frac{\pi\beta}{b}\,d\alpha\,d\beta.$$

Thus a relationship is found between the deflection, the initial imperfection, the temperature, the load and the elastic characteristics of the plate being heated.

Examination of equation (4.7) shows that in the case of temperature variations of a flexible plate with initial imperfection there may occur a "snap" phenomenon at a constant load.

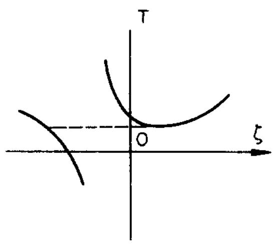

FIG. 1.

To illustrate this phenomenon, Fig. 1 presents a graph showing the relation $\zeta = \zeta(T)$ at certain fixed values of ζ_0 and $Z = \text{const}$. In calculations it is assumed that the elastic characteristics of the plate material vary with temperature in such a manner that $(1-\nu_1\nu_2)/E_1 = (1+\lambda T)/E_0$, where $\lambda = \text{const}$.

5. PARAMETRIC VIBRATION OF A FLEXIBLE PLATE

Let a flexible, isotropic ($E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$) rectangular ($a \times b$) plate ($k_1 = k_2 = 0$, $A = B = 1$), hinge-supported throughout the contour, be placed in a temperature field

$$T = T(\gamma, t) = T(-\gamma, t)$$

under the effect of the high-frequency tangential load

$$X = P\cos\theta t$$

applied at the edges $\alpha = 0$ and $\alpha = a$.

On the basis of the assumptions, developed in Section 1, let us consider parametric vibrations of the plate, when [10, 12]

$$E = E_0 - eT, \qquad T = \frac{T_{\max}(\gamma)}{t_1}\, t, \tag{5.1}$$

whence

$$E = E_0 - e_1 t \qquad e_1 = \frac{eT_{\max}(\gamma)}{t_1}, \tag{5.2}$$

where E_0 denotes the modulus of elasticity at initial temperature, e is a coefficient, determined from experiments.

To simplify the presentation, the temperature variations of Poisson's ratio ν are neglected.

For the problem under consideration we obtain from (1.11) the following initial system of nonlinear differential equations of dynamic stability

$$\begin{gathered}(D_0 - D_1 t)\Delta^2 w - Q_4(w, \varphi) + P\cos\theta t\frac{\partial^2 w}{\partial\alpha^2} + \rho h\frac{\partial^2 w}{\partial t^2} = 0,\\ \frac{1}{c_0 - c_1 t}\Delta^2\varphi + \tfrac{1}{2}Q_4(w, w) = 0,\end{gathered} \tag{5.3}$$

where

$$c_0 = E_0 h, \qquad c_1 = 2\int_0^{h/2} e_1\, d\gamma,$$

$$D_0 = \frac{E_0 h^3}{12(1-\nu^2)},\ D_1 = 2\int_0^{h/2} e_1\gamma^2\, d\gamma, \qquad \Delta = \frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2},$$

ρ denotes the shell material density.

The assumption

$$\begin{aligned} w &= f(t)\sin\lambda_n\alpha\sin\mu_m\beta,\\ \varphi &= F(t)\sin\lambda_n\alpha\sin\mu_m\beta,\\ \lambda_n &= n\pi/a, \qquad \mu_m = m\pi/b,\end{aligned}$$

identically satisfies the conditions of the hinge-supported edges of the plate.

Applying the variational method [14] to the nonlinear system (5.3), we obtain

$$\begin{gathered}\Gamma(t, f) \equiv \ddot{f} + \omega^2(t)[1 - 2\mu(t)\cos\theta t]f + \chi(t)f^3 = 0,\\ F = -\frac{16}{3}\frac{\lambda_n\mu_m}{ab}\frac{c_0 - c_1 t}{\lambda_n^2 + \mu_m^2}\dot{f}^2,\end{gathered} \tag{5.4}$$

where

$$\begin{gathered}\omega^2(t) = \omega_0^2 - \Omega t, \qquad \omega_0^2 = \frac{D_0}{\rho h}(\lambda_n^2 + \mu_m^2)^2, \qquad \Omega = \frac{D_1}{\rho h}(\lambda_n^2 + \mu_m^2)^2,\\ \mu(t) = \frac{P\lambda_n^2}{2(D_0 - D_1 t)(\lambda_n^2 + \mu_m^2)^2},\end{gathered} \tag{5.5}$$

$$\chi = \chi_0 - \chi_1 t, \qquad \chi_i = \frac{512 c_i}{9\rho h a^2 b^2}\cdot\frac{\lambda_n^2\mu_m^2}{(\lambda_n^2 + \mu_m^2)^2} \qquad (i = 0, 1). \tag{5.6}$$

Here $\omega(t)$ denotes the "proper frequency" of the plate under heating, ω_0 is the "proper frequency" of the plate at initial temperature, $\mu(t)$ is the excitation coefficient.

It can be approximately assumed that at the time of passing through the main parametric resonance the plate under heating during one j-th period vibrates following the law

$$f(t) = b_{jk} l_k(t) \qquad (k = 1, 2), \tag{5.7}$$

where b_{jk} are the unknown amplitudes of resonance vibrations,

$$l_1(t) = \cos\frac{\theta t}{2}, \qquad l_2(t) = \sin\frac{\theta t}{2}, \tag{5.8}$$

and $k = 1, 2$ indicates the lower and the upper boundaries of the unstable range, respectively.

The equations for determination of the amplitudes b_{jk} by the method of Bubnov–Galerkin assume the form

$$\int_{(4\pi/\theta)(j-1)}^{(4\pi/\theta)j} \Gamma[t, b_{jk} l_k(t)] l_k(t)\, dt = 0 \qquad (k = 1, 2). \tag{5.9}$$

Considering, that

$$\omega^2(t)\mu(t) = \frac{P\lambda_n^2}{2\rho h} = \text{const}, \qquad j = \frac{t_j \theta}{4\pi} \tag{5.10}$$

(t_j is the time, necessary to reach the j-th vibration period of the plate under heating), from (5.9), by virtue of (5.8), we obtain

$$b_{jk}^2(t_j) = \frac{1}{3}\left[\chi(t_j) + 2\chi_1 \frac{\pi}{\theta}\right]^{-1} \left[\theta^2 - \bar{\theta}_{*k}^2 - 8\Omega \frac{\pi}{\theta}\right], \tag{5.11}$$

where for critical frequencies of the main parametric resonance of the plate under heating $\bar{\theta}_{*k}$, determined without considering the effect of temperature dynamics, the following relation is obtained

$$\bar{\theta}_{*k}^2(t_j) = 4\omega^2(t_j)[1 + (-1)^k \mu(t_j)] \qquad (k = 1, 2). \tag{5.12}$$

Examination of formulae (5.5), (5.6) and (5.12) shows that the values $\chi(t)$ and $\bar{\theta}_{*k}(t)$ decrease with time, which increases the amplitude of resonance vibrations of the plate under heating. The effect of temperature dynamics is characterized by the last additive terms in the square brackets of the formula (5.11), that is $2\chi_1(\pi/\theta)$ and $8\Omega(\pi/\theta)$, and it decreases the amplitude of resonance vibration of the plate under heating.

From the condition $b_{jk} = 0$ the equation to determine the critical frequencies of the principal parametric resonance of the plate under heating is obtained

$$\theta^3 - \bar{\theta}_{*k}^2 \theta - 8\pi\Omega = 0. \tag{5.13}$$

The free term in equation (5.13), characterizing the effect of temperature dynamics, increases the values of critical frequencies of the principal parametric resonance.

Let us also consider the case when the time of a considerable variation of the elasticity modulus is sufficiently great. In this case, neglecting terms characterizing the effect of temperature dynamics, from (5.11) the relation is obtained

$$b_{jk}^2(t) = \frac{\theta^2 - \bar{\theta}_{*k}^2(t_j)}{3\chi(t_j)}$$

and from the condition $b_{jk} = 0$ we have

$$\theta^2 - \bar{\theta}_{*k}^2(t_j) = 0.$$

Thus, in terms of temperature the problem appears to be quasi-static, that is in this case in order to determine the resonance vibration amplitudes and the critical frequencies of parametric resonance of the non-heated plate it is necessary in the known relations to substitute c_1 and D_1 for the rigidities $c_1(t)$ and $D_1(t)$.

Without repeating the calculations, the results can be shown for the case, when instead of the relations (5.1) and (5.2) we have

$$E = E_0 - eT, \qquad T = \frac{T_{\max}(\gamma)}{t_1^2}t^2, \qquad E = E_0 - e_2 t^2,$$

where

$$e_2 = \frac{eT_{\max}(\gamma)}{t_1^2}.$$

For the amplitudes of resonance vibrations of the plate under heating the formula is obtained

$$b_{jk}^2(t_j) = \frac{1}{3}\left[\chi(t_j) + \frac{4\pi}{\theta}\chi_2\left(t_j - \frac{4\pi}{3\theta}\right)\right]^{-1}\left[\theta^2 - \bar{\theta}_{*k}^2 - \frac{16\pi}{\theta}\bar{\Omega}\left(t_j - \frac{4\pi}{3\theta}\right)\right], \tag{5.14}$$

where

$$\bar{\Omega} = \frac{1}{\rho h}D_2(\lambda_n^2 + \mu_m^2)^2, \qquad \chi_2 = \frac{512c_2}{9\rho h a^2 b^2}\frac{\lambda_n^2\mu_m^2}{(\lambda_n^2 + \mu_m^2)^2},$$

$$c_2 = 2\int_0^{h/2} e_2\,d\gamma, \qquad D_2 = 2\int_0^{h/2} e_2\gamma^2\,d\gamma.$$

Examination of the formula (5.14) shows that in this case the terms, characterizing the effect of temperature dynamics, depend upon time.

For example, let us consider the plate, having the following characteristics

$$a = b = \pi\,\text{m}, \qquad h = 10^{-2}\,\text{m}, \qquad E_0 = 2{\cdot}10^{10}\,\text{kg/m}^2, \qquad \nu = 0{,}408,$$

$$\rho = 10^3\,\text{kg sec}^2/\text{m}^4, \qquad e_1 = \frac{E_0}{2t_1}, \qquad P = 2000\,\text{kg/m}.$$

In the case $m = n = 1$, $t_1 = 1$ sec we have

$$\theta_{*01}^2 = 2800\,1/\text{sec}^2, \qquad \theta_{*02}^2 = 3600\,1/\text{sec}^2, \tag{5.15}$$

$$\bar{\theta}_{*1}^2(1) = 1200\,1/\text{sec}^2, \qquad \bar{\theta}_{*2}^2(1) = 2000\,1/\text{sec}^2, \tag{5.16}$$

$$\theta_{*1}^2(1) = 1463\,1/\text{sec}^2, \qquad \theta_{*2}^2(1) = 2214\,1/\text{sec}^2. \tag{5.17}$$

Frequencies (5.15) indicate the range of principal parametric resonance of the plate under consideration at the initial temperature, those of (5.16)—at quasi-static (in temperature sense) formulation of the problem, those of (5.17) considering the temperature dynamics effect.

Thus, the consideration of the temperature dynamics influence with an essential change of the material elasticity modulus in a rather short interval of time leads to a certain increase of the values of critical frequencies and to a decrease of width of the range of principal parametric resonance.

In conclusion we shall note, that the method of Mandelshtam [15] leads to similar results.

6. STABILITY OF AN INFINITELY LONG CYLINDRICAL SHELL IN SUPERSONIC FLOW OF A COMPRESSIBLE GAS

Let us consider an isotropic ($E_1 = E_2 = E(t)$, $\nu_1 = \nu_2 = \nu$, $\alpha_1 = \alpha_2 = \alpha_0$) infinitely long circular cylindrical shell ($R_1 = \infty, R_2 = R$), in supersonic flow of a gas with the undisturbed velocity U, directed along the shell generatrix.

Along with the previously introduced assumptions, when determining the aerodynamic pressure, "the law of flat sections" (piston theory) [16] is assumed to be valid.

Excluding the nonlinear terms from the system of equations (1.11) and assuming $A = 1$, $B = 1$, the following is obtained for the problem considered

$$D(t)\Delta\Delta w + \frac{1}{R}\frac{\partial^2 \varphi}{\partial \alpha^2} = Z - \rho_0 h \frac{\partial^2 w}{\partial t^2},$$
$$\Delta\Delta\varphi - \frac{E(t)h}{R}\frac{\partial^2 w}{\partial \alpha^2} = 0, \tag{6.1}$$

where

$$\Delta = \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2}, \qquad D(t) = \frac{E(t)h^3}{12(1-\nu^2)}$$

is the cylindrical rigidity of the shell, ρ_0 stands for the shell material density.

In the problem under consideration for Z we have [8]

$$Z = -2\rho_0 h\varepsilon \frac{\partial w}{\partial t} + \nabla p. \tag{6.2}$$

Here ε is the coefficient of linear attenuation, ∇p is the excessive gas pressure, which according to the accepted assumptions, has the form

$$\nabla p = -\frac{\varkappa p_\infty}{a_\infty}\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial \alpha}\right), \tag{6.3}$$

where p_∞ is the pressure of undisturbed gas flow, a_∞ is the velocity of sound for undisturbed flow, $\varkappa$ is the polytropic curve index.

Making use of (6.2) and (6.3), from (6.1) we obtain

$$D(t)\Delta^2\Delta^2 w + \frac{E(t)h}{R^2}\frac{\partial^4 w}{\partial \alpha^4} + \left[\rho_0 h\frac{\partial^2}{\partial t^2} + \left(2\rho_0 h\varepsilon + \frac{\varkappa p_\infty}{a_\infty}\right)\frac{\partial}{\partial t} + \frac{\varkappa p_\infty}{a_\infty}U\frac{\partial}{\partial \alpha}\right]\Delta^2 w = 0. \tag{6.4}$$

We will seek the solution of the equation (6.4) in the form

$$w = W(t)e^{-ik\alpha}\cos\frac{n\beta}{R}, \tag{6.5}$$

where $W(t)$ is a complex function of the real argument, $k = \pi/\lambda$ is the wave number, λ is the semi-wave length in the direction of the generatrix, n is the number of waves along the shell directrix.

Substituting (6.5) into (6.4), in order to determine $W(t)$, we will obtain

$$\frac{d^2 W}{dt^2} + (2\varepsilon + \mu)\frac{dW}{dt} + \frac{D(t)}{\rho_0 h R^4}\left[(m^2 + n^2)^2 + \frac{12R^2(1 - \nu^2)}{h^2}\frac{m^4}{(m^2 + n^2)^2}\right]W - i\frac{m\mu}{R}UW = 0, \qquad (6.6)$$

where

$$\mu = \frac{\varkappa p_\infty}{\rho_0 h a_\infty}, \qquad m = kR. \qquad (6.7)$$

By substitution

$$W(t) = f(t)\exp[-(\varepsilon + \mu/2)t] \qquad (6.8)$$

from (6.6) we obtain

$$\ddot{f}(t) + \delta^2\psi(t)f(t) = 0, \qquad (6.9)$$

where

$$\delta^2 = \frac{E_0}{12\gamma_0 h(1 - \nu^2)},$$

$$\psi(t) = \frac{E(t)}{E_0}\frac{h^3 g}{R^4}\left[(m^2 + n^2)^2 + \frac{12(1 - \nu^2)R^2}{h^2}\frac{m^4}{(m^2 + n^2)^2}\right] - \frac{12\gamma_0 h(1 - \nu^2)}{E_0}\left[i\frac{m\mu}{R}U + (\varepsilon + \mu/2)^2\right], \qquad (6.10)$$

E_0 is the initial modulus of elasticity, γ_0 is the specific gravity of the shell material, g is the acceleration of gravity.

Since δ is a large parameter, the equation (6.9), as above, can be solved by the method of asymptotic integration [1, 2] that is the solution will be sought in the form

$$f(t) = \Phi(t;\delta)e^{\delta\omega(t)},$$
$$\Phi(t;\delta) = \Phi_0(t) + \delta^{-1}\Phi_1(t) + \delta^{-2}\Phi_2(t) + \ldots \qquad (6.11)$$

Without going into the details already known and restricting ourselves to the first approximation of asymptotic integration we finally obtain for $W(t)$

$$W(t) = \psi^{-\frac{1}{4}}[C_1 e^{i\eta(t) - (\varepsilon + \mu/2)t} + C_2 e^{-i\eta(t) - (\varepsilon + \mu/2)t}], \qquad (6.12)$$

where

$$\eta(t) = \delta\int_0^t \sqrt{[\psi(t)]}\,dt = \eta_1(t) + i\eta_2(t), \qquad (6.13)$$

$C_1 = a_1 + ib_1$, $C_2 = a_2 + ib_2$ are the arbitrary complex constants.

Examining (6.12), it should be noted, that the nature of the $W(t)$ function depends upon both the velocity of the flow U, and the variation of the modulus of elasticity $E(T)$.

If the modulus of amplitude $W(t)$ increases with time, then it may conditionally be considered that the shell is in an unstable state. Otherwise the state of shell is stable.

The flow velocity U_{cr}, at which $\mathrm{d}/\mathrm{d}t|W(t)| = 0$, will be referred to as critical.

For the sake of definiteness it should be assumed here that

$$E(t) = E_0 - E_0^* T, \qquad T = \frac{T_{max}}{t_1} t \qquad (0 \le t \le t_1)$$

or

$$E(t) = E_0 - E_1 t \qquad \text{where} \quad E_1 = E_0^* T_{max} t_1^{-1}.$$

Then from (6.12) and (6.13) we will obtain

$$W(t) = r^{-\frac{1}{4}} e^{-\eta_2(t) - (\varepsilon + \mu/2)t} [C_1 e^{i\eta_1(t)} + C_2 e^{-i\eta_1(t)} e^{2\eta_2(t)}] \left(\cos \frac{\vartheta + 2\pi j}{4} - i \sin \frac{\vartheta + 2\pi j}{4}\right), \tag{6.14}$$

$$\eta(t) = -\frac{2\delta}{3s}\left[r^{\frac{3}{2}}\left(\cos\frac{3\vartheta + 2\xi\pi}{2} - i\sin\frac{3\vartheta + 2\xi\pi}{2}\right) - r_0^{\frac{3}{2}}\left(\cos\frac{3\vartheta_0 + 2\xi\pi}{2} - i\sin\frac{3\vartheta_0 + 2\xi\pi}{2}\right)\right] \quad (j = 0, 1, 2, 3;\ \xi = 0, 1), \tag{6.15}$$

$$\begin{aligned} \eta_1(t) &= -\frac{2\delta}{3s}\left[r^{\frac{3}{2}}(t)\cos\frac{3\vartheta + 2\xi\pi}{2} - r_0^{\frac{3}{2}}\cos\frac{3\vartheta_0 + 2\xi\pi}{2}\right], \\ \eta_2(t) &= \frac{2\delta}{3s}\left[r^{\frac{3}{2}}(t)\sin\frac{3\vartheta + 2\xi\pi}{2} - r_0^{\frac{3}{2}}\sin\frac{3\vartheta_0 + 2\xi\pi}{2}\right], \end{aligned} \tag{6.16}$$

where

$$r^2(t) = \frac{1}{E_0^2 R^2}[B^2(t) + G^2 U^2], \qquad r_0 = r(0),$$

$$\vartheta(t) = \operatorname{arc\,tg}\frac{GU}{B(t)}, \qquad \vartheta_0 = \vartheta(0),$$

$$S = \frac{E_1}{E_0}\frac{h^3 g}{R^4}\left[(m^2+n^2)^2 + \frac{12R^2(1-\nu^2)}{h^2}\frac{m^4}{(m^2+n^2)^2}\right],$$

$$B(t) = \frac{E_0 - E_1 t}{E_1} E_0 R s - 12\gamma_0 h R(1-\nu^2)(\varepsilon + \mu/2)^2, \qquad G = 12\gamma_0 h(1-\nu^2) m\mu.$$

As it has been shown above, the critical velocity is found from the condition†

$$\frac{\mathrm{d}}{\mathrm{d}t}|W(t)| = 0, \tag{6.17}$$

which in an expanded form is written as follows

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\{r^{-\frac{1}{4}} e^{-\eta_2(t) - (\varepsilon+\mu/2)t}[(e^{2\eta_2(t)}[a_2\cos\eta_1(t) + b_2\sin\eta_1(t)] + a_1\cos\eta_1(t) - b_1\sin\eta_1(t))^2 \\ + (e^{2\eta_2(t)}[b_2\cos\eta_1(t) - a_2\sin\eta_1(t)] + b_1\cos\eta_1(t) + a_1\sin\eta_1(t))^2]^{\frac{1}{2}}\} = 0. \end{aligned} \tag{6.18}$$

† It is easy to prove that the cases $\xi = 0$ and $\xi = 1$ give the same results for the critical velocity of flutter. Therefore only the case $\xi = 0$ is dealt with later.

Due to the bulky appearance of the equation (6.18) the analytical determination of U_{cr} is rather difficult. However, considering the fact that the modulus of the first additive term of the function $W(t)$, that is

$$|C_1|r^{-\frac{1}{4}}\exp[\eta_2(t)-(\varepsilon+\mu/2)t],$$

decreases with time regardless of the flow velocity, the critical velocity can be approximately determined from the condition (6.17), imposed only on the second additive term of the function $W(t)$, that is from the condition

$$\frac{d}{dt}\{r^{-\frac{1}{4}}\exp[-\eta_2(t)-(\varepsilon+\mu/2)t]\} = 0. \tag{6.19}$$

In view of

$$B(t) > 0, \qquad \frac{d\eta_2(t)}{dt} = -\delta r^{\frac{1}{4}}\sin\frac{\vartheta}{2}, \tag{6.20}$$

from (6.15) and (6.16) we find

$$0 \le \vartheta \le \pi/2 \qquad \text{and} \quad \eta_2(t) \le 0 \qquad (0 \le t \le t_1).$$

Then from (6.19) we obtain for the determination of the critical velocity

$$\frac{1}{4}\frac{E_0RsB(t)}{B^2(t)+G^2U^2} = -\frac{1}{\sqrt{[24\gamma_0hR(1-\nu^2)]}}\sqrt{\{\sqrt{[B^2(t)+G^2U^2]}-B(t)\}}+\varepsilon+\mu/2. \tag{6.21}$$

In most cases $B^2(t) \gg G^2U^2$, then from (6.21) we may obtain

$$\frac{U_{cr}}{U_{cr}^*} = 1 - \frac{1}{4(\varepsilon+\mu/2)}\frac{E_1}{E_0 - E_1t}, \tag{6.22}$$

where

$$U_{cr}^* = \frac{1}{R}\sqrt{\left(\frac{D(t)}{\rho_0h}\right)}\left[\frac{(m^2+n^2)^2}{m^2} + \frac{12R^2(1-\nu^2)}{h^2}\frac{m^2}{(m^2+n^2)^2}\right]^{\frac{1}{4}}\left(1+\frac{2\varepsilon}{\mu}\right).$$

is the critical velocity, found by quasi-static theory from the view point of temperature, that is when in the final formula of the critical velocity for the shell being heated, $E_0 - E_1t$ is taken instead of E.

It is easy to see from (6.22) that the dynamic critical velocity U_{cr} is less than the critical velocity, found by quasi-static theory, and in the case of a small attenuation the former may considerably differ from the latter.

In the case of the shell not being heated ($E_1 = 0$), the familiar formula for the critical velocity of flutter [17] is obtained from (6.21)

$$U_{cr} = \frac{1}{R}\sqrt{\left(\frac{D_0}{\rho_0h}\right)}\left[\frac{(m^2+n^2)^2}{m^2} + \frac{12R^2(1-\nu^2)}{h^2}\frac{m^2}{(m^2+n^2)^2}\right]^{\frac{1}{4}}\left(1+\frac{2\varepsilon}{\mu}\right).$$

The results obtained may be extended to the case of an infinite plate. To do this it is sufficient to set in the final results

$$R = \infty, \qquad n = \infty, \qquad \frac{n}{R} = \frac{\pi}{\lambda_\beta}, \qquad \frac{m}{R} = \frac{\pi}{\lambda_\alpha}, \tag{6.23}$$

where λ_α and λ_β denote the semi-wave lengths in α and β directions, respectively.

Substituting (6.23) into (6.21) and passing to the limit, for determination of the critical velocity of the infinite plate the equation is obtained

$$\frac{1}{4}\frac{E_0 S B(t)}{B^2(t) + G^2 U^2} = -\frac{1}{\sqrt{[24\gamma_0 h(1 - \nu^2)]}}\sqrt{\{\sqrt{[B^2(t) + G^2U^2]} - B(t)\}} + (\varepsilon + \mu/2), \tag{6.24}$$

where, along with the adopted notation, one has

$$B(t) = \frac{E_0 - E_1 t}{E_1} E_0 S - 12\gamma_0 h(1 - \nu^2)(\varepsilon + \mu/2)^2,$$

$$G = 12\gamma_0 h(1 - \nu^2)\frac{\pi^\mu}{\lambda_\alpha}, \qquad S = \frac{E_1}{E_0} h^3 g\left(\frac{\pi^2}{\lambda_\alpha^2} + \frac{\pi^2}{\lambda_\beta^2}\right)^2.$$

In particular, when $B^2(t) \gg G^2U^2$, the formula (6.22) is obtained from (6.24), but in this case it should be assumed that

$$U_{cr}^* = \sqrt{\left[\frac{D(t)}{\rho_0 h}\right]}\left(\frac{\pi}{\lambda_\alpha} + \frac{\pi\lambda_\alpha}{\lambda_\beta^2}\right)\left(1 + \frac{2\varepsilon}{\mu}\right).$$

$$m_0 = \rho_0 h = 16\times10^{-6}\,\mathrm{kg\ sec^2/cm^3}, \qquad \epsilon = 0.113\ \mathrm{sec}^{-1}$$

$$\bar{U}_{cr} = \frac{\mu\sqrt{[12\rho_0(1-\nu^2)]}}{\sqrt{E_0 2\epsilon h\left(\frac{\pi}{\lambda_x} + \frac{\pi\lambda_x}{\lambda_y^2}\right)}} U_{cr}, \qquad \bar{U}_{cr}^0 = U_{cr}^*(E_1 = 0)$$

$$P = -\frac{\chi P_\infty}{a_\infty} U \frac{\partial w}{\partial \alpha}$$

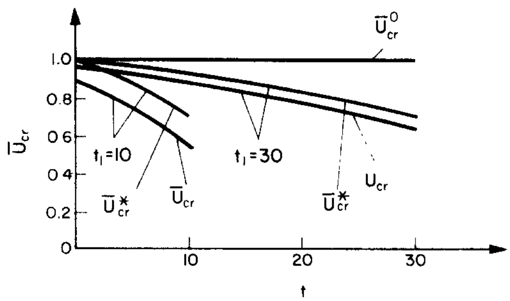

FIG. 2.

Figure 2 presents the dependence of the critical velocity on time t.

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Résumé—Dans ce texte ont été considérés les problèmes des vibrations libres et forcées, de la stabilité statique et dynamique et de l'aéroélasticité des enveloppes et plaques orthotropiques placées dans un champ de température variable. Il a été considéré que les caractéristiques physico-mécaniques de la matière de l'enveloppe (plaque) dépendent de la température.

Il a été indiqué qu'en tenant compte de la dépendance, sur la température, des propriétés physico-mécaniques de la matière de l'enveloppe (plaque) des changements qualitatifs et quantitatifs essentiels sont introduits dans le problème de la vibration et de la stabilité.

Zusammenfassung—Probleme von freien und erzwungenen Schwingungen statischer und dynamischer Stabilität und Aeroelastizität von orthotropischen Schalen und Platten die sich in einem veränderlichen Temperaturbereich befinden werden erwogen. Es wird angenommen, dass die physikalisch-mechanische Charakteristik des Materials der Schale (Platte) von der Temperatur abhängig ist. Es wird gezeigt, dass, wenn die Abhängigkeit der physikalischmechanischen Eigenschaften des Materials der Schale (Platte) von der Temperatur in Rechnung genommen wird, es zu wichtigen qualitativen und quantitativen Abänderungen in dem Problem von Schwingung und Stabilität führt.

Резюме—Рассматриваются задачи свободных и вынужденных колебаний, статической и динамической устойчивости и аэроупругости ортотропных оболочек и пластин, находящихся в переменном температурном поле. При этом принимается, что физико-механические характеристики материала оболочки (пластины) зависят от температуры нагрева.

Показывается, что учет изменения физико-механических свойств материала оболочки (пластины) в зависимости от температуры вносит существенные качественные и количественные изменения в задачи колебаний и устойчивости.